FACTORIZED DOMAIN WALL PARTITION FUNCTIONS IN TRIGONOMETRIC VERTEX MODELS

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Abstract. We obtain factorized domain wall partition functions for two sets of trigonometric vertex models: **1.** The N-state Deguchi-Akutsu models, for $N \in \{2,3,4\}$ (and conjecture the result for all $N \geq 5$), and **2.** The $sl(r+1|s+1)$ Perk-Schultz models, for $\{r,s \in \mathbb{N}\}$, where (given the symmetries of these models) the result is independent of $\{r,s\}$.

0. Introduction

Domain wall partition functions (DWPF's) were first proposed and evaluated in determinant form, for the spin-$\frac{1}{2}$ vertex model[1] on a finite square lattice, in [1, 2]. At the free fermion point of the spin-$\frac{1}{2}$ model, this determinant is in Cauchy form and therefore factorizes. More recently, determinant expressions for the DWPF's of spin-$\frac{N-1}{2}$ models and also of level-1 affine $so(N)$ models (for certain discrete values of the crossing parameter) were obtained in [3] and in [4], respectively.

State variable conjugation. We are interested in models with state variables $\{\sigma\}$. Each state variable takes discrete integral values, $\sigma \in \{1, \cdots, N\}$. We define *'state variable conjugation'* as replacing each state variable σ by $(N - \sigma + 1)$. The models mentioned above are invariant under this conjugation.

The N-state Deguchi-Akutsu (N–DA) models, $N \geq 2$ [5], are models with vertex weights[2], that depend on two sets of parameters: **1.** Vertical and horizontal rapidities, and **2.** Vertical and horizontal external field variables. They reduce in the limit of no external fields to the spin-$\frac{N-1}{2}$ models at their respective free fermion points.

Factorized DWPF. In [6], a determinant expression for the DWPF of the 2–DA model was obtained using the arguments of [1,2], but only for zero values of the rapidities. This determinant is in Cauchy form and therefore factorizes. For general values of all parameters, no determinant expression was found, and it was argued on general grounds that no such expression exists. However, using the F–basis of [7], a factorized expression for the 2–DA DWPF was obtained.

In this work, we extend the above result to the N-DA models, $N \in \{2,3,4\}$. Our results are restricted to $N \in \{2,3,4\}$ because our proofs require the explicit expressions of the weights, while the number of vertices grows $\sim O(N^3)$. However, our results are quite simple and have a uniform dependence on N, which allows us to conjecture that our expression extends to all $N \geq 2$.

Non-invariance under state variable conjugation. Our proofs rely on the non-invariance of the N-DA models under state variable conjugation (for non-vanishing external fields). This leads us to look for other models that are similarly non-invariant.

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[1]In the sequel, *'model'* will always mean *'trigonometric vertex model'*.

[2]In the sequel, *'weight'* will always stand for *'vertex weight'*.

The $sl(r+1|s+1)$ **Perk-Schultz (PS) models**, $\{r, s \in \mathbb{N}\}$ [8], form another class of models that are non-invariant under state variable conjugation, in this case because the state variables belong to two different sets with different statistics.

In these models, the definition of domain wall boundary conditions (DWBC's) is not unique. From experience with the N-DA models, we propose a definition that leads to factorized DWPF's. The symmetries of the Perk-Schultz models are such that the result is independent of $\{r, s\}$.

Outline of paper. In section **1**, we recall basic definitions, introduce the N–DA models and obtain the corresponding factorized DWPF. In section **2**, we do the same for the $sl(r+1|s+1)$ Perk-Schultz models. Section **3** contains brief remarks and appendix **A** lists the weights of the N–DA models, for $N \in \{2, 3, 4\}$.

1. The N-state Deguchi-Akutsu (N–DA) models

1.1. **The lattice.** We work on a square lattice consisting of L vertical and L horizontal lines, label the vertical lines from left to right and the horizontal from top to bottom.

We assign the i-th vertical line an orientation from bottom to top, a complex rapidity variable u_i and a complex external field variable α_i. We assign the j-th horizontal line an orientation from left to right, a complex rapidity variable v_j and a complex external field variable β_j.

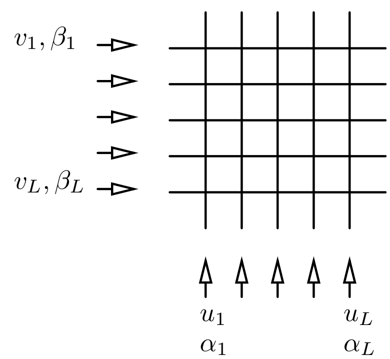

Figure 1. *An $L \times L$ square lattice, with oriented lines and variables.*

1.2. **Vertices.** Each lattice line intersects with L other lines. A line segment between two intersections is a bond. To each bond, we assign a state variable $\sigma \in \{1, 2, \cdots, N\}$. The intersection of the i-th vertical line and the j-th horizontal line, together with the four bonds adjacent to it, and the set of state variables on these bonds, is a vertex v_{ij}.

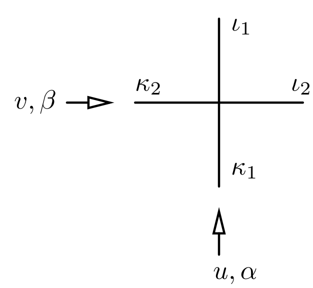

Figure 2. *The vertex corresponding to $X_{\alpha,\beta}(u-v)^{\iota_1,\iota_2}_{\kappa_2,\kappa_1}$.*

1.3. **Weights.** To each vertex v_{ij} we assign a weight w_{ij}, that depends on the state variables on the four bonds of that vertex, the difference of rapidity variables flowing through the vertex, and the two external field variables flowing through the vertex. Specifically, a vertex with vertical rapidity and external field variable $\{u, \alpha\}$, horizontal rapidity and external field variable $\{v, \beta\}$, and state variables $\{\iota_1, \iota_2, \kappa_1, \kappa_2\}$ is assigned the weight $X_{\alpha,\beta}(u-v)^{\iota_1,\iota_2}_{\kappa_2,\kappa_1}$. These weights satisfy the Yang-Baxter equation:

$$\sum_{\lambda_1,\lambda_2,\lambda_3} X_{\alpha,\beta}(u-v)^{\iota_1,\iota_2}_{\lambda_2,\lambda_1} X_{\alpha,\gamma}(u-w)^{\lambda_1,\iota_3}_{\lambda_3,\kappa_1} X_{\beta,\gamma}(v-w)^{\lambda_2,\lambda_3}_{\kappa_3,\kappa_2} = \sum_{\lambda_1,\lambda_2,\lambda_3} X_{\beta,\gamma}(v-w)^{\iota_2,\iota_3}_{\lambda_3,\lambda_2} X_{\alpha,\gamma}(u-w)^{\iota_1,\lambda_3}_{\kappa_3,\lambda_1} X_{\alpha,\beta}(u-v)^{\lambda_1,\lambda_2}_{\kappa_2,\kappa_1} \tag{1}$$

Expressions for all $X_{\alpha,\beta}(u-v)^{\iota_1,\iota_2}_{\kappa_2,\kappa_1}$ are given in [5], for $N \in \{2,3,4\}$. For completeness we include them in appendix **A**. From these expressions, one can check that the weights of the N–DA model are not invariant under conjugating the state variables. This is due to the presence of external fields $\{\alpha,\beta\}$.

Switching off the external fields restores the symmetry of the weights (up to global gauge transformations). The (symmetrized) no-field weights coincide with the weights of the spin-$\frac{N-1}{2}$ models at their free fermion point. This can be checked trivially for $N=2$ by setting $\alpha=\beta=\sqrt{-1}$.

1.4. **Minimal and maximal state variables.** We refer to the state variable $\sigma = 1$ as minimal, and to $\sigma = N$ as maximal.

1.5. **The c_+ vertex.** We refer to the unique vertex with minimal state variables incoming from the left and exiting from above, maximal state variables incoming from below and exiting from the right, as shown in Figure **3**, as the c_+ vertex. In the N–DA models $c_+(\alpha,\beta,u-v) = X_{\alpha,\beta}(u-v)^{1,N}_{1,N}$.

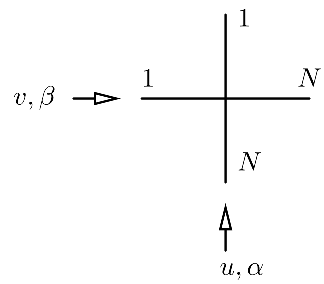

Figure 3. *The c_+ vertex.*

1.6. **Domain wall boundary conditions (DWBC).** We define DWBC's in the N–DA model as a form of *expanded c_+ vertex*: All (boundary) bonds on the left and top carry minimal state variables, while all bonds on the right and below carry maximal state variables.

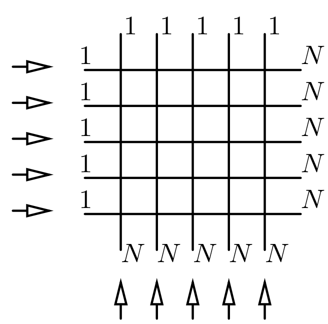

Figure 4. *N–DA DWBC's.*

1.7. **Line-permuting vertices.** Any model, on a finite lattice with DWBC's, has a pair of vertices that can be used to permute adjacent lattice lines, as we will see in detail below. We call these vertices $\{a_+, a_-\}$. In the N–DA models,

$$\begin{aligned} a_+(\alpha,\beta,u-v) &= X_{\alpha,\beta}(u-v)^{1,1}_{1,1} \\ a_-(\alpha,\beta,u-v) &= X_{\alpha,\beta}(u-v)^{N,N}_{N,N} \end{aligned} \tag{2}$$

1.8. **Domain wall partition function (DWPF).** As always, the DWPF on an $L \times L$ lattice, $Z^{DA}_{L\times L}$, is the sum over all weighted configurations that satisfy the DWBC. The weight of each configuration is the product of the weights of the vertices:

$$Z^{DA}_{L\times L} = \sum_{\text{configurations}} \left(\prod_{\text{vertices}} w_{ij} \right) \tag{3}$$

1.9. **Properties of the N–DA DWPF.** The N–DA DWPF, $N \in \{2,3,4\}$, satisfies four properties:

Property 1. From the DWBC's, $Z^{DA}_{L\times L}$ has the form

$$Z^{DA}_{L\times L} \Big(\{\alpha\}, \{\beta\}, \{u\}, \{v\} \Big) = e^{(N-1)u_1} p \Big(\{\alpha\}, \{\beta\}, \{u\}, \{v\} \Big) \tag{4}$$

where p is a polynomial of degree $(L-1)(N-1)$ in e^{u_1}.

This can be seen from the fact that the rapidity u_1 only appears in the left-most column, and every vertex in that column is of the form $X_{\alpha_1,\beta_j}(u_1 - v_j)^{\iota,\kappa}_{1,\lambda}$. From appendix **A**, one can check that

$$X_{\alpha_1,\beta_j}(u_1 - v_j)^{\iota,\kappa}_{1,\lambda} = e^{(\lambda-\iota)u_1} q(\alpha_1, \beta_j, u_1, v_j)^{\iota,\kappa}_{1,\lambda} \tag{5}$$

where $q^{\iota,\kappa}_{1,\lambda}$ is a polynomial of degree $(N-1+\iota-\lambda)$ in e^{u_1}. Property **1** then follows by noticing that every lattice configuration in the DWPF receives a contribution of

$$X_{\alpha_1,\beta_1}(u_1 - v_1)^{1,\kappa_1}_{1,\lambda_1} X_{\alpha_1,\beta_2}(u_1 - v_2)^{\lambda_1,\kappa_2}_{1,\lambda_2} \ldots X_{\alpha_1,\beta_L}(u_1 - v_L)^{\lambda_{L-1},\kappa_L}_{1,N} \tag{6}$$

from the left-most column.

Property 2. $Z^{DA}_{L\times L}$ has zeros in e^{u_1} at the $(L-1)(N-1)$ points

$$e^{u_1} = \frac{e^{u_k}}{\rho^{j-1}\alpha_1\alpha_k}, \quad j \in \{1, \ldots, N-1\}, k \in \{2, \ldots, L\} \tag{7}$$

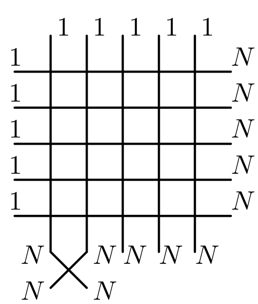

Figure 5. *Attaching an a_- vertex.*

This can be seen as follows: Multiply $Z^{DA}_{L\times L} \Big(\{\alpha\}, \{\beta\}, \{u\}, \{v\} \Big)$ with the vertex $a_-(\alpha_1, \alpha_2, u_1 - u_2)$. This corresponds to attaching a type a_- vertex from below, as in Figure **5**.

Using the Yang-Baxter equation, we slide the inserted vertex through the lattice until it emerges from the top, as in Figure **6**.

The inserted $a_-(\alpha_1, \alpha_2, u_1 - u_2)$ vertex emerges as an $a_+(\alpha_1, \alpha_2, u_1 - u_2)$ vertex, and in the process, the two left-most vertical lattice lines are permuted.

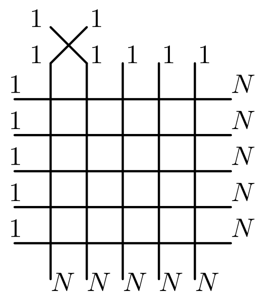

Figure 6. *Extracting an a_+ vertex.*

We conclude that

$$Z^{DA}_{L\times L}\left(\{\alpha\},\{\beta\},\{u\},\{v\}\right)=\frac{a_+(\alpha_1,\alpha_2,u_1-u_2)}{a_-(\alpha_1,\alpha_2,u_1-u_2)}\times \\ Z^{DA}_{L\times L}\left(\{\alpha_2,\alpha_1,\ldots\},\{\beta\},\{u_2,u_1,\ldots\},\{v\}\right) \tag{8}$$

Iterating the above procedure $(L-1)$ times, we obtain

$$Z^{DA}_{L\times L}\left(\{\alpha\},\{\beta\},\{u\},\{v\}\right)=\prod_{j=2}^{L}\left(\frac{a_+(\alpha_1,\alpha_j,u_1-u_j)}{a_-(\alpha_1,\alpha_j,u_1-u_j)}\right)\times \\ Z^{DA}_{L\times L}\left(\{\alpha_2,\ldots,\alpha_L,\alpha_1\},\{\beta\},\{u_2,\ldots,u_L,u_1\},\{v\}\right) \tag{9}$$

The locations of the $(L-1)(N-1)$ zeros in e^{u_1} follow from Equation **9**.

Property 3. $Z^{DA}_{L\times L}$ obeys the recursion relation

$$Z^{DA}_{L\times L}\Big|_{e^{u_1}=\frac{\beta_L}{\alpha_1}e^{v_L}}=\left(\frac{\beta_L}{\alpha_1}\right)^{N-1}\prod_{j=1}^{N-1}\left(\sqrt{1-\rho^{j-1}\alpha_1^2}\sqrt{1-\rho^{j-1}\beta_L^2}\right)\times \\ \prod_{j=1}^{N-1}\prod_{k=1}^{L-1}\left(1-\rho^{j-1}\beta_L\beta_k e^{v_L-v_k}\right)\prod_{k=2}^{L}\left(e^{u_k-v_L}-\rho^{j-1}\alpha_k\beta_L\right)Z^{DA,(1L)}_{(L-1)\times(L-1)} \tag{10}$$

where $Z^{DA,(1L)}_{(L-1)\times(L-1)}$ is the DWPF on an $(L-1)\times(L-1)$ lattice, with the omission of external field variables $\{\alpha_1,\beta_L\}$ and rapidities $\{u_1,v_L\}$. This is seen by noting the lower-left vertex must be $X_{\alpha_1,\beta_L}(u_1-v_L)^{\iota,\kappa}_{1,N}$, which, as can be verified in appendix **A**, satisfies

$$X_{\alpha_1,\beta_L}(u_1-v_L)^{\iota,\kappa}_{1,N}\Big|_{e^{u_1}=\frac{\beta_L}{\alpha_1}e^{v_L}}=0,\quad\text{unless }\iota=1,\ \kappa=N \tag{11}$$

Hence, setting $e^{u_1}=\frac{\beta_L}{\alpha_1}e^{v_L}$ in $Z^{DA}_{L\times L}$ freezes the lower-left vertex to a type $c+$, the remainder of the bottom row to type a_-, and the remainder of the left-most column to type a_+. Equation **10** follows from these considerations.

Property 4. The DWPF on a 1×1 lattice is given by the c_+ vertex

$$Z^{DA}_{1\times 1} = e^{(N-1)(u_1-v_1)} \prod_{j=1}^{N-1} \left(\sqrt{1-\rho^{j-1}\alpha_1^2}\sqrt{1-\rho^{j-1}\beta_1^2} \right) \tag{12}$$

which follows from the definition of the DWBC and the weights.

Lemma 1. The above four properties determine the N–DA DWPF, $N \in \{2,3,4\}$, uniquely.

Proof. Write $Z^{DA}_{L\times L}$ for the actual DWPF, and assume there exists some other $\Pi^{DA}_{L\times L}$ which satisfies all of the preceding four properties. By Property **4**, we have $Z^{DA}_{1\times 1} = \Pi^{DA}_{1\times 1}$, which is the basis for induction. Fix an integer $n \geq 2$. From Properties **1** and **2**, $\Pi^{DA}_{n\times n}$ must be equal to $Z^{DA}_{n\times n}$, up to a multiplicative term, $\mathcal{C}$, that does not depend on e^{u_1}.

From Property **3** and the inductive assumption $Z^{DA}_{(n-1)\times(n-1)} = \Pi^{DA}_{(n-1)\times(n-1)}$, we find that the multiplicative constant $\mathcal{C} = 1$. Hence, $Z^{DA}_{n\times n} = \Pi^{DA}_{n\times n}$, proving the uniqueness claim by induction.

1.10. **Evaluation of the N–DA DWPF, $N \in \{2,3,4\}$.** We postulate an expression for $Z^{DA}_{L\times L}$, then show that it satisfies the four properties of the previous section.

Lemma 2.

$$\boxed{\begin{aligned} Z^{DA}_{L\times L} = \prod_{j=1}^{L} &\left(e^{(N-1)j(u_j-v_j)} \prod_{k=1}^{N-1} \sqrt{1-\rho^{k-1}\alpha_j^2}\sqrt{1-\rho^{k-1}\beta_j^2} \right) \times \\ &\prod_{1\leq i<j\leq L} \prod_{k=1}^{N-1} \left(1-\rho^{k-1}\alpha_i\alpha_j e^{u_i-u_j} \right) \left(1-\rho^{k-1}\beta_j\beta_i e^{v_j-v_i} \right) \qquad (13) \end{aligned}}$$

Proof. By inspection, the product expression in Equation **13** satisfies Property **1** and **4**. It contains a factor of $\prod_{j=2}^{L}\prod_{k=1}^{N-1}\left(1-\rho^{k-1}\alpha_1\alpha_j e^{u_1-u_j}\right)$, which means it possesses the $(L-1)(N-1)$ zeros required by Property **2**. Finally, working directly from Equation **13**, we obtain

$$\begin{aligned} Z^{DA}_{L\times L} = e^{-L(N-1)v_L} \prod_{k=1}^{L} \left(e^{(N-1)u_k} \right) \prod_{j=1}^{N-1} \left(\sqrt{1-\rho^{j-1}\alpha_1^2}\sqrt{1-\rho^{j-1}\beta_L^2} \right) \times \\ \prod_{k=1}^{N-1}\prod_{i=1}^{L-1} \left(1-\rho^{k-1}\beta_L\beta_i e^{v_L-v_i} \right) \prod_{j=2}^{L} \left(1-\rho^{k-1}\alpha_1\alpha_j e^{u_1-u_j} \right) Z^{DA,(1L)}_{(L-1)\times(L-1)} \end{aligned} \tag{14}$$

Evaluating Equation **14** at the point $e^{u_1} = \frac{\beta_L}{\alpha_1} e^{v_L}$, we obtain

$$
\begin{aligned}
&Z^{DA}_{L\times L}\Big|_{e^{u_1}=\frac{\beta_L}{\alpha_1}e^{v_L}} = \left(\frac{\beta_L}{\alpha_1}\right)^{N-1} e^{-(L-1)(N-1)v_L} \times \\
&\prod_{k=2}^{L}\left(e^{(N-1)u_k}\right)\prod_{j=1}^{N-1}\left(\sqrt{1-\rho^{j-1}\alpha_1^2}\sqrt{1-\rho^{j-1}\beta_L^2}\right)\times \\
&\prod_{k=1}^{N-1}\prod_{i=1}^{L-1}\left(1-\rho^{k-1}\beta_L\beta_i e^{v_L-v_i}\right)\prod_{j=2}^{L}\left(1-\rho^{k-1}\beta_L\alpha_j e^{v_L-u_j}\right) Z^{DA,(1L)}_{(L-1)\times(L-1)}
\end{aligned}
\tag{15}
$$

Rearranging factors in Equation **15**, one recovers Equation **10** in Property **3**, as required. This concludes our proof that the N–DA DWPF, $N \in \{2,3,4\}$, factorizes. □

Conjecture. Equation **13** is valid for all $N \geq 2$. This conjecture is based on the fact that our results for $N \in \{2,3,4\}$ have a uniform dependence on N, in which only the c_+ and line-permuting vertices appear. A study of these very vertices for a few values of $N > 4$, indicates that they have analogous forms and properties, hence our conjecture. However, a proof of this conjecture requires detailed knowledge of all vertices, for all $N > 4$, which is beyond the scope of this work.

2. The $sl(r+1|s+1)$ Perk-Schultz (PS) models

This section will be brief, as the arguments are the same as for the N-DA models. The PS models are defined on the same lattice, with the same orientations as the N–DA models. The differences are in the weights, as those of the PS models do not depend on external field variables.

2.1. Two sets of state variables.

Following [9], we define two sets $B_-=\{1,\ldots,s+1\}$, $B_+=\{s+2,\ldots,r+s+2\}$, and their union

$$
B = \{\underbrace{1,\ldots,s+1}_{B_-},\underbrace{s+2,\ldots,r+s+2=N}_{B_+}\}
$$

2.2. The weights.

Let $a, b \in B$. The non-vanishing weights of the $sl(r+1|s+1)$ Perk-Schultz (PS) models are

$$
R^{a,a}_{a,a}(u) = \begin{cases} \frac{\sinh\eta(1-u)}{\sinh\eta}, & a\in B_- \\ \frac{\sinh\eta(1+u)}{\sinh\eta}, & a\in B_+ \end{cases}
$$

$$
R^{a,b}_{b,a}(u) = \begin{cases} -\frac{\sinh\eta u}{\sinh\eta}, & a,b\in B_- \text{ or } a,b\in B_+ \\ \frac{\sinh\eta u}{\sinh\eta}, & \text{otherwise} \end{cases}
$$

$$
R^{a,b}_{a,b}(u) = \begin{cases} e^{+\eta u}, & a<b \\ e^{-\eta u}, & a>b \end{cases} \tag{16}
$$

where η is a crossing parameter. The labelling of the vertices follows the same convention as in Figure **2**.

2.3. Non-invariance under state variable conjugation.

It is clear, by inspection, that the weights of the $sl(r+1|s+1)$ PS model are not invariant under conjugation of state variables, where $N = r+s+2$.

2.4. **Symmetries and $\{r, s\}$-independence.** From Equations **16**, it is clear that the PS weights are symmetric in $\sigma_- \in B_-$ and (separately) in the $\sigma_+ \in B_+$[3]. One can also see that, choosing any $\sigma_- \in B_-$ and any $\sigma_+ \in B_+$, as domain wall boundary variables, no other state variables appear in the domain wall configurations, and one obtains a factorized DWPF that is independent of $\{r, s\}$, in other words, the same result one obtains in the $sl(1|1)$ model[4].

2.5. **PS Domain wall boundary conditions.** We define the $sl(r+1|s+1)$ PS DWBC as follows: The state variables on all bonds on the right and lower boundaries are maximal, $\sigma = N = r + s + 2 \in B_+$, and the state variables on all bonds on the left and upper boundaries are minimal, $\sigma = 1 \in B_-$.

Notice that, using the symmetries of the weights, we could have taken *any* state variable $\sigma \in B_-$ on the left and top, and any state variable $\sigma \in B_+$ on the right and below. The advantage of the above choice is that the labels of the c_+ and line-permuting vertices are precisely the same as those in the N–DA models.

2.6. **Properties of the** $sl(r+1|s+1)$ **PS DWPF.** The $sl(r+1|s+1)$ PS DWPF, $Z^{PS}_{L\times L}$, satisfies the following four properties, the proofs of which are precisely analogous to those of the N–DA models.

Property 1. Given the PS weights and DWBC's, and writing $U_1 = e^{\eta u_1}$, $Z^{PS}_{L\times L}$ has the form

$$Z^{PS}_{L\times L}\left(\{u\},\{v\}\right) = U_1^{-L+2}\, p\left(\{u\},\{v\}\right) \tag{17}$$

where p is a polynomial of degree $(L-1)$ in U_1^2.

Property 2. Using the line-permuting vertices, $\{R^{1,1}_{1,1}, R^{N,N}_{N,N}\}$, and the Yang-Baxter equations, it is straightforward to show that

$$Z^{PS}_{L\times L}\left(\{u\},\{v\}\right) = \prod_{j=2}^{L} \frac{R^{1,1}_{1,1}(u_1-u_j)}{R^{N,N}_{N,N}(u_1-u_j)} Z^{PS}_{L\times L}\left(u_2,\ldots,u_L,u_1,\{v\}\right)$$

which gives the $(L-1)$ zeros of p.

Property 3. Setting $u_1 = v_L$, we freeze the lower left-hand corner to an $R^{1,N}_{1,N}$, and obtain the recursion relation

$$\begin{aligned} &Z^{PS}_{L\times L}|_{u_1=v_L} = \\ &R^{1,N}_{1,N}(0)\left(\prod_{j=1}^{L-1} R^{1,1}_{1,1}(v_L-v_j)\right)\left(\prod_{j=2}^{L} R^{N,N}_{N,N}(u_j-v_L)\right) Z^{PS,(1L)}_{(L-1)\times(L-1)} \end{aligned} \tag{18}$$

Property 4. The initial condition is given by the c_+ vertex

$$Z^{PS}_{1\times 1}(u_1,v_1) = R^{1,N}_{1,N}(u_1-v_1)$$

Lemma 3. The above four properties determine the PS DWPF uniquely.

Proof. The proof is identical to that of Lemma **1**.

[3] $sl(N)$ Belavin models are analogously symmetric, but all state variables take values in one set only $\sigma \in \{1,\cdots,N\}$.

[4] In $sl(N)$ Belavin models, choosing any two distinct state variables to impose DWBC's, no other state variables appear in the configurations, and one obtains Izergin's determinant expression for the spin-$\frac{1}{2}$ model, which corresponds to $N = 2$.

2.7. **Evaluation of the PS DWPF.** We postulate an expression for $Z^{PS}_{L\times L}$, then show that it satisfies the four properties of the previous section.

Lemma 4.

$$\boxed{\begin{aligned} Z^{PS}_{L\times L}&\Big(\{u\},\{v\}\Big) = \\ &\left(\prod_{k=1}^{L} R^{1,N}_{1,N}(u_k - v_k)\right)\left(\prod_{1\le i<j\le L} R^{1,1}_{1,1}(u_i-u_j)R^{1,1}_{1,1}(v_j-v_i)\right) \end{aligned}} \tag{19}$$

Proof. The proof is identical to that of Lemma **2** in the case of the N-DA models, as one can show that the expression in Equation **19** obeys the required four properties[5].

3. Remarks

Domain wall partition functions are important in physics because of their role in the algebraic Bethe ansatz approach to correlation functions [11], and in combinatorics, because in the special case of the spin-$\frac{1}{2}$ model, they lead to counting alternating sign matrices [12].

The factorizing DWPF's discussed in this work do not lead to new combinatorics: One can define corresponding combinatorial objects, but the weights do not lead to 1-counting (they cannot all be simultaneously set to 1). The factorization of the DWPF's (probably) reflects the fermionic nature of the underlying models and the absence of interesting counting is in turn a reflection of that nature.

On the other hand, factorized DWPF's are easier to handle than determinants and one expects (from experience with the spin-$\frac{1}{2}$ model at the free fermion point) that computing the corresponding correlation functions will be easier than in generic models.

The main point of this work is to observe that, if the weights of the line-permuting vertices have different zeros, then the corresponding DWPF's factorize. The models discussed in this paper offer trigonometric vertex examples of this observation. It is possible that all such models are either fermionic (with or without external fields) as in the N–DA models, or contain fermions, as in the $sl(r+1|s+1)$ PS models.

Appendix A

In the following, $w = u - v$, $x = e^w$ and $\rho = e^{\frac{2\pi i n}{N}}$, where $\{n, N\}$ are co-prime. There is no crossing parameter because these models are external field deformations of spin-$\frac{N-1}{2}$ models with a crossing parameter set to the free fermion value.

N=2. (6 vertices)

$$\begin{aligned} X_{\alpha,\beta}(w)^{1,1}_{1,1} &= (1-\alpha\beta x) & X_{\alpha,\beta}(w)^{1,2}_{1,2} &= x\sqrt{(1-\alpha^2)(1-\beta^2)} \\ X_{\alpha,\beta}(w)^{1,2}_{2,1} &= (\alpha-\beta x) & X_{\alpha,\beta}(w)^{2,1}_{1,2} &= (\beta-\alpha x) \\ X_{\alpha,\beta}(w)^{2,1}_{2,1} &= \sqrt{(1-\alpha^2)(1-\beta^2)} & X_{\alpha,\beta}(w)^{2,2}_{2,2} &= (x-\alpha\beta) \end{aligned}$$

[5] In [10], a determinant expression for the DWPF of the $sl(1|1)$ PS model was obtained. It is straightforward to show that it evaluates to the above product expression.

N=3. (19 vertices)

$$
\begin{aligned}
X_{\alpha,\beta}(w)^{1,1}_{1,1} &= (1-\alpha\beta x)(1-\alpha\beta\rho x) \\
X_{\alpha,\beta}(w)^{1,2}_{1,2} &= x\sqrt{(1-\alpha^2)(1-\beta^2)}(1-\alpha\beta\rho x) \\
X_{\alpha,\beta}(w)^{1,2}_{2,1} &= (\alpha-\beta x)(1-\alpha\beta\rho x) \\
X_{\alpha,\beta}(w)^{1,3}_{1,3} &= x^2\sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\beta^2)(1-\beta^2\rho)} \\
X_{\alpha,\beta}(w)^{1,3}_{2,2} &= \sqrt{(1-\alpha^2)(1-\beta^2\rho)}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\rho}}x(\alpha-\beta x) \\
X_{\alpha,\beta}(w)^{1,3}_{3,1} &= (\alpha-\beta x)(\alpha-\beta\rho x) \\
X_{\alpha,\beta}(w)^{2,1}_{1,2} &= (\beta-\alpha x)(1-\alpha\beta\rho x) \\
X_{\alpha,\beta}(w)^{2,1}_{2,1} &= \sqrt{(1-\alpha^2)(1-\beta^2)}(1-\alpha\beta\rho x) \\
X_{\alpha,\beta}(w)^{2,2}_{1,3} &= \sqrt{(1-\alpha^2\rho)(1-\beta^2)}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\rho}}x(\beta-\alpha x) \\
X_{\alpha,\beta}(w)^{2,2}_{2,2} &= (1-\alpha^2)(1-\beta^2\rho)x-(\beta-\alpha x)(\beta x-\alpha\rho) \\
X_{\alpha,\beta}(w)^{2,2}_{3,1} &= \sqrt{(1-\alpha^2)(1-\beta^2\rho)}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\rho}}(\alpha-\beta x) \\
X_{\alpha,\beta}(w)^{2,3}_{2,3} &= x(x-\alpha\beta)\sqrt{(1-\alpha^2\rho)(1-\beta^2\rho)} \\
X_{\alpha,\beta}(w)^{2,3}_{3,2} &= (1+\rho)(\alpha-\beta x)(x-\alpha\beta) \\
X_{\alpha,\beta}(w)^{3,1}_{1,3} &= (\beta-\alpha x)(\beta-\alpha\rho x) \\
X_{\alpha,\beta}(w)^{3,1}_{2,2} &= \sqrt{(1-\beta^2)(1-\alpha^2\rho)}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\rho}}(\beta-\alpha x) \\
X_{\alpha,\beta}(w)^{3,1}_{3,1} &= \sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\beta^2)(1-\beta^2\rho)} \\
X_{\alpha,\beta}(w)^{3,2}_{2,3} &= (1+\rho)(\beta-x\alpha)(x-\alpha\beta) \\
X_{\alpha,\beta}(w)^{3,2}_{3,2} &= \sqrt{(1-\alpha^2\rho)(1-\beta^2\rho)}(x-\alpha\beta) \\
X_{\alpha,\beta}(w)^{3,3}_{3,3} &= (x-\alpha\beta)(x-\alpha\beta\rho)
\end{aligned}
$$

N=4. (44 vertices)

$$
\begin{aligned}
X_{\alpha,\beta}(w)^{1,1}_{1,1} &= (1-\alpha\beta x)(1-\alpha\beta\rho x)(1-\alpha\beta\rho^2 x) \\
X_{\alpha,\beta}(w)^{1,2}_{1,2} &= x\sqrt{(1-\alpha^2)(1-\beta^2)}(1-\alpha\beta\rho x)(1-\alpha\beta\rho^2 x) \\
X_{\alpha,\beta}(w)^{1,2}_{2,1} &= (\alpha-\beta x)(1-\alpha\beta\rho x)(1-\alpha\beta\rho^2 x) \\
X_{\alpha,\beta}(w)^{1,3}_{1,3} &= x^2\sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\beta^2)(1-\beta^2\rho)}(1-\alpha\beta\rho^2 x) \\
X_{\alpha,\beta}(w)^{1,3}_{2,2} &= \sqrt{(1-\alpha^2)(1-\beta^2\rho)}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\rho}}x(\alpha-\beta x)(1-\alpha\beta\rho^2 x) \\
X_{\alpha,\beta}(w)^{1,3}_{3,1} &= (\alpha-\beta x)(\alpha-\beta\rho x)(1-\alpha\beta\rho^2 x) \\
X_{\alpha,\beta}(w)^{1,4}_{1,4} &= x^3\sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\alpha^2\rho^2)}\sqrt{(1-\beta^2)(1-\beta^2\rho)(1-\beta^2\rho^2)} \\
X_{\alpha,\beta}(w)^{1,4}_{2,3} &= \sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\beta^2)(1-\beta^2\rho)}\frac{\sqrt{1-\rho^3}}{\sqrt{1-\rho}}x^2(\alpha-\beta x) \\
X_{\alpha,\beta}(w)^{1,4}_{3,2} &= \sqrt{(1-\alpha^2)(1-\beta^2\rho^2)}\frac{\sqrt{1-\rho^3}}{\sqrt{1-\rho}}x(\alpha-\beta x)(\alpha-\beta\rho x)
\end{aligned}
$$

$$X_{\alpha,\beta}(w)^{1,4}_{4,1} = (\alpha-\beta x)(\alpha-\beta\rho x)(\alpha-\beta\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{2,1}_{1,2} = (\beta-\alpha x)(1-\alpha\beta\rho x)(1-\alpha\beta\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{2,1}_{2,1} = \sqrt{(1-\alpha^2)(1-\beta^2)}(1-\alpha\beta\rho x)(1-\alpha\beta\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{2,2}_{1,3} = \sqrt{(1-\alpha^2\rho)(1-\beta^2)}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\rho}}x(\beta-\alpha x)(1-\alpha\beta\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{2,2}_{2,2} = ((1-\alpha^2)(1-\beta^2\rho)x-(\beta-\alpha x)(\beta x-\alpha\rho))(1-\alpha\beta\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{2,2}_{3,1} = \sqrt{(1-\alpha^2)(1-\beta^2\rho)}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\rho}}(\alpha-\beta x)(1-\alpha\beta\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{2,3}_{1,4} = \sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\beta^2)(1-\beta^2\rho)}\frac{\sqrt{1-\rho^3}}{\sqrt{1-\rho}}(\alpha-\beta x)(1-\alpha\beta\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{2,3}_{2,3} = \sqrt{(1-\alpha^2\rho)(1-\beta^2\rho)}((1-\beta^2)(1-\alpha^2\rho^2)-(1+\rho)x(\alpha x-\beta\rho)(\alpha-\beta x))$$
$$X_{\alpha,\beta}(w)^{2,3}_{3,2} = (\alpha-\beta x)((1-\alpha^2\beta^2)(1-\rho^3)x-\rho(\alpha x-\beta\rho)(\alpha-\beta x))$$
$$X_{\alpha,\beta}(w)^{2,3}_{4,1} = \frac{\sqrt{1-\rho^3}}{\sqrt{1-\rho}}\sqrt{(1-\alpha^2)(1-\beta^2\rho^2)}(\alpha-\beta x)(\alpha-\beta\rho x)$$
$$X_{\alpha,\beta}(w)^{2,4}_{2,4} = x^2\sqrt{1-\alpha^2\rho}\sqrt{(1-\alpha^2\rho^2)(1-\beta^2\rho)(1-\beta^2\rho^2)}(x-\alpha\beta)$$
$$X_{\alpha,\beta}(w)^{2,4}_{3,3} = x\frac{\sqrt{(1-\rho^2)(1-\rho^3)}}{1-\rho}\sqrt{(1-\alpha^2\rho)(1-\beta^2\rho^2)}(x-\alpha\beta)(\alpha-\beta x)$$
$$X_{\alpha,\beta}(w)^{2,4}_{4,2} = \frac{(1-\rho^3)(x-\alpha\beta)(\alpha-\beta x)(\alpha-\beta\rho x)}{1-\rho}$$
$$X_{\alpha,\beta}(w)^{3,1}_{1,3} = (\beta-\alpha x)(\beta-\alpha\rho x)(1-\alpha\beta\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{3,1}_{2,2} = \sqrt{(1-\beta^2)(1-\alpha^2\rho)}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\rho}}(\beta-\alpha x)(1-\alpha\beta\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{3,1}_{3,1} = \sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\beta^2)(1-\beta^2\rho)}(1-\alpha\beta\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{3,2}_{1,4} = \sqrt{(1-\alpha^2\rho^2)(1-\beta^2)}\frac{\sqrt{1-\rho^3}}{\sqrt{1-\rho}}x(\beta-\alpha x)(\beta-\alpha\rho x)$$
$$X_{\alpha,\beta}(w)^{3,2}_{2,3} = (\beta-\alpha x)((1-\alpha^2\beta^2)(1-\rho^3)x-\rho(\beta-\alpha x)(\beta x-\alpha\rho))$$
$$X_{\alpha,\beta}(w)^{3,2}_{3,2} = \sqrt{(1-\alpha^2\rho)(1-\beta^2\rho)}((1-\alpha^2)(1-\beta^2\rho^2)x-(1+\rho)(\beta-\alpha x)(\beta x-\alpha\rho))$$
$$X_{\alpha,\beta}(w)^{3,2}_{4,1} = \sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\beta^2\rho)(1-\beta^2\rho^2)}\frac{\sqrt{1-\rho^3}}{\sqrt{1-\rho}}(\alpha-\beta x)$$
$$X_{\alpha,\beta}(w)^{3,3}_{2,4} = x\frac{\sqrt{(1-\rho^2)(1-\rho^3)}}{1-\rho}\sqrt{(1-\alpha^2\rho^2)(1-\beta^2\rho)}(x-\alpha\beta)(\alpha-\beta x)$$
$$X_{\alpha,\beta}(w)^{3,3}_{3,3} = ((1-\alpha^2\rho)(1-\beta^2\rho^2)x-(1+\rho+\rho^2)(\beta-\alpha x)(\beta x-\alpha\rho))(x-\alpha\beta)$$
$$X_{\alpha,\beta}(w)^{3,3}_{4,2} = \sqrt{(1-\alpha^2\rho)(1-\beta^2\rho)}\frac{\sqrt{(1-\rho^2)(1-\rho^3)}}{1-\rho}(x-\alpha\beta)(\alpha-\beta x)$$
$$X_{\alpha,\beta}(w)^{3,4}_{3,4} = x\sqrt{1-\alpha^2\rho^2}\sqrt{1-\beta^2\rho^2}(x-\alpha\beta)(x-\alpha\beta\rho)$$
$$X_{\alpha,\beta}(w)^{3,4}_{4,3} = \frac{1-\rho^3}{1-\rho}(x-\alpha\beta)(x-\alpha\beta\rho)(\alpha-\beta x)$$
$$X_{\alpha,\beta}(w)^{4,1}_{1,4} = (\beta-\alpha x)(\beta-\alpha\rho x)(\beta-\alpha\rho^2 x)$$
$$X_{\alpha,\beta}(w)^{4,1}_{2,3} = \sqrt{(1-\alpha^2\rho^2)(1-\beta^2)}\frac{\sqrt{1-\rho^3}}{\sqrt{1-\rho}}(\beta-\alpha x)(\alpha-\beta x)$$

$$X_{\alpha,\beta}(w)_{3,2}^{4,1} = \sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\beta^2\rho)(1-\beta^2\rho^2)}\frac{\sqrt{1-\rho^3}}{\sqrt{1-\rho}}(\beta-\alpha x)$$
$$X_{\alpha,\beta}(w)_{4,1}^{4,1} = \sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\alpha^2\rho^2)}\sqrt{(1-\beta^2)(1-\beta^2\rho)(1-\beta^2\rho^2)}$$
$$X_{\alpha,\beta}(w)_{2,4}^{4,2} = \frac{1-\rho^3}{1-\rho}(x-\alpha\beta)(\beta-\alpha x)(\beta-\alpha\rho x)$$
$$X_{\alpha,\beta}(w)_{3,3}^{4,2} = \sqrt{(1-\alpha^2\rho^2)(1-\beta^2\rho)}\frac{\sqrt{(1-\rho^2)(1-\rho^3)}}{1-\rho}(x-\alpha\beta)(\beta-\alpha x)$$
$$X_{\alpha,\beta}(w)_{4,2}^{4,2} = \sqrt{(1-\alpha^2)(1-\alpha^2\rho)(1-\beta^2\rho)(1-\beta^2\rho^2)}(x-\alpha\beta)$$
$$X_{\alpha,\beta}(w)_{3,4}^{4,3} = \frac{1-\rho^3}{1-\rho}(x-\alpha\beta)(x-\alpha\beta\rho)(\beta-\alpha x)$$
$$X_{\alpha,\beta}(w)_{4,3}^{4,3} = \sqrt{(1-\alpha^2\rho^2)(1-\beta^2\rho^2)}(x-\alpha\beta)(x-\alpha\beta\rho)$$
$$X_{\alpha,\beta}(w)_{4,4}^{4,4} = (x-\alpha\beta)(x-\alpha\beta\rho)(x-\alpha\beta\rho^2)$$

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